# ALGEBRAIC $K$-THEORY OF GROUPS WREATH PRODUCT WITH FINITE GROUPS

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ABSTRACT. The Farrell-Jones Fibered Isomorphism Conjecture for the stable topological pseudoisotopy theory has been proved for several classes of groups. For example for discrete subgroups of Lie groups ([4]), virtually poly-infinite cyclic groups ([4]), Artin braid groups ([6]), a class of virtually poly-surface groups ([14]) and virtually solvable linear group ([5]). We extend these results in the sense that if $G$ is a group from the above classes then we prove the conjecture for the wreath product $G \wr H$ for $H$ a finite group. The need for this kind of extension is already evident in [6], [13] and [14]. We also prove the conjecture for some other classes of groups.

## 1. INTRODUCTION

In this article we are mainly concerned about the Fibered Isomorphism Conjecture for the stable topological pseudoisotopy theory. We extend some existing results and also prove the conjecture for some other classes of groups. Finally we deduce a corollary for the Isomorphism Conjecture for the algebraic $K$-theory in dimension $\leq 1$ (see Corollary 2.3).

Before we state the theorem let us recall that given two groups $G$ and $H$ the *wreath product* $G \wr H$ is by definition the semidirect product $G^H \rtimes H$ where the action of $H$ on $G^H$ is the regular action.

**Theorem 1.1.** *The Fibered Isomorphism Conjecture for the stable topological pseudoisotopy theory is true for the group $G \wr H$ where $H$ is a finite group and $G$ is one of the following groups.*

*(a). Virtually polycyclic groups.*

*(b). Virtually solvable subgroups of $GL_n(\mathbb{C})$.*

  *(c). Cocompact discrete subgroups of virtually connected Lie groups.*
  *(d). Artin full braid groups.*
  *(e). Weak strongly poly-surface groups (see Definition 4.1).*
  *(f). Extensions of closed surface groups by surface groups.*
  *(g). $\pi_1(M) \rtimes \mathbb{Z}$, where $M$ is a closed Seifert fibered space.*

*Proof.* See Corollary 2.2 and Remark 2.1.                              □

(In the notation defined in Definition 2.1 the Theorem says that the $\mathrm{FIC}^{P\mathcal{H}^?_*}_{\mathcal{VC}}$ is true for $G \wr H$ or equivalently the $\mathrm{FICwF}^{P\mathcal{H}^?_*}_{\mathcal{VC}}$ is true for $G$.)

The $\mathrm{FICwF}^{P\mathcal{H}^?_*}_{\mathcal{VC}}$ was proved for 3-manifold groups in [13] and [14] and for the fundamental groups of a certain class of graphs of virtually poly-cyclic groups in [15].

For the extended Fibered Isomorphism Conjecture $\mathrm{FICwF}^{P\mathcal{H}^?_*}_{\mathcal{VC}}$ we deduce the following proposition using $(a)$ of Lemma 3.4. This result is not yet known if we replace $\mathrm{FICwF}^{P\mathcal{H}^?_*}_{\mathcal{VC}}$ by $\mathrm{FIC}^{P\mathcal{H}^?_*}_{\mathcal{VC}}$ (see 5.4.5 in [9] for the question).

**Proposition 1.1.** *Let $G$ be a group containing a finite index subgroup $\Gamma$. Assume that the $\mathrm{FICwF}^{P\mathcal{H}^?_*}_{\mathcal{VC}}$ is true for $\Gamma$. Then the $\mathrm{FICwF}^{P\mathcal{H}^?_*}_{\mathcal{VC}}$ is true for $G$.*

We work in the general setting of the conjecture in equivariant homology theory (see [1]). We find out a set of properties which are all satisfied in the pseudoisotopy case of the conjecture. And assuming these properties we prove a theorem (Theorem 2.2) for the Isomorphism Conjecture in equivariant homology theory. Theorem 1.1 is then a particular case of Theorem 2.2. We mention in Corollary 2.3 another consequence of our result for the Isomorphism Conjecture in the algebraic $K$-theory case. We also hope that the general Theorem 2.2 will be useful for future application.

The methods we use in this paper were developed in [15].

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## 2. Statement of the general Theorem

We first recall the general statement of the Isomorphism Conjecture in equivariant homology theory from [1] and also recall some definitions from [15].

A class $\mathcal{C}$ of subgroups of a group $G$ is called a *family of subgroups* if $\mathcal{C}$ is closed under taking subgroups and conjugations. For a family of subgroups $\mathcal{C}$ of $G$, $E_{\mathcal{C}}(G)$ denotes a $G$-CW complex so that the

fixpoint set $E_{\mathcal{C}}(G)^H$ is contractible if $H \in \mathcal{C}$ and empty otherwise. And $R$ denotes an associative ring with unit.

**(Fibered) Isomorphism Conjecture:** (see definition 1.1 in [1]) Let $\mathcal{H}_*^?$ be an equivariant homology theory with values in $R$-modules. Let $G$ be a group and $\mathcal{C}$ be a family of subgroups of $G$. Then the *Isomorphism Conjecture* for the pair $(G, \mathcal{C})$ states that the projection $p : E_{\mathcal{C}}(G) \to pt$ to the point $pt$ induces an isomorphism

$$\mathcal{H}_n^G(p) : \mathcal{H}_n^G(E_{\mathcal{C}}(G)) \simeq \mathcal{H}_n^G(pt)$$

for $n \in \mathbb{Z}$. And the *Isomorphism Conjecture in dimension $\leq k$*, for $k \in \mathbb{Z}$, states that $\mathcal{H}_n^G(p)$ is an isomorphism for $n \leq k$. Finally the *Fibered Isomorphism Conjecture* for the pair $(G, \mathcal{C})$ states that given a group homomorphism $\phi : K \to G$ the Isomorphism Conjecture is true for the pair $(K, \phi^*\mathcal{C})$. Here $\phi^*\mathcal{C} = \{H < K \mid \phi(H) \in \mathcal{C}\}$.

From now on let $\mathcal{C}$ be a class of groups which is closed under isomorphisms, taking subgroups and taking quotients. The class $\mathcal{VC}$ of all virtually cyclic groups has these properties. Given a group $G$ we denote by $\mathcal{C}(G)$ the class of subgroups of $G$ belonging to $\mathcal{C}$. Then $\mathcal{C}(G)$ is a family of subgroups of $G$ which is also closed under taking quotients.

**Definition 2.1.** *(see definition 2.1 in* [15]*)* If the (Fibered) Isomorphism Conjecture is true for the pair $(G, \mathcal{C}(G))$ we say that the *(F)IC$_{\mathcal{C}}^{\mathcal{H}_*^?}$ is true for $G$* or simply say *(F)IC$_{\mathcal{C}}^{\mathcal{H}_*^?}(G)$ is satisfied*. Also we say that the *(F)ICwF$_{\mathcal{C}}^{\mathcal{H}_*^?}(G)$ is satisfied* if the *(F)IC$_{\mathcal{C}}^{\mathcal{H}_*^?}$* is true for $G \wr H$ for all finite groups $H$.

Let us denote by $_P\mathcal{H}_*^?$ and $_K\mathcal{H}_*^?$ the equivariant homology theories arise in the Isomorphism Conjecture of Farrell and Jones ([4]) corresponding to the stable topological pseudoisotopy theory and the algebraic $K$-theory respectively. For these homology theories the conjecture is identical with the conjecture made in §1.6 and §1.7 in [4]. (See sections 5 and 6 in [1] for the second case and 4.2.1 and 4.2.2 in [9] for the first case.)

Note that if the FIC$_{\mathcal{C}}^{\mathcal{H}_*^?}$ (respectively FICwF$_{\mathcal{C}}^{\mathcal{H}_*^?}$) is true for a group $G$ then the FIC$_{\mathcal{C}}^{\mathcal{H}_*^?}$ (respectively FICwF$_{\mathcal{C}}^{\mathcal{H}_*^?}$) is true for subgroups of $G$. We refer to this fact as the *hereditary property*. Also note that the (F)IC$_{\mathcal{C}}^{\mathcal{H}_*^?}$ is true for $H \in \mathcal{C}$.

**Definition 2.2.** *(see definition 2.2 in* [15]*)* We say that the $\mathcal{P}_{\mathcal{C}}^{\mathcal{H}_*^?}$-*property* is satisfied if for $G_1, G_2 \in \mathcal{C}$ the product $G_1 \times G_2$ satisfies the FIC$_{\mathcal{C}}^{\mathcal{H}_*^?}$.

In the following the notation $A \rtimes B$ stands for the semidirect product of $A$ by $B$ with respect to some arbitrary action of $B$ on $A$.

**Definition 2.3.** We define the following notations.

$\mathcal{N}$ : The FICwF$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for $\pi_1(M)$ for closed nonpositively curved Riemannian manifolds $M$.

$\mathcal{B}$ : The FICwF$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for $\mathbb{Z}^n \rtimes \mathbb{Z}$ for all $n$.

$\mathcal{L}$ : If $G = lim_{i \to \infty} G_i$ and the FIC$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for $G_i$ for each $i$ then the FIC$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for $G$.

**Theorem 2.1.** *$\mathcal{B}$, $\mathcal{N}$ and $\mathcal{L}$ are satisfied for the FICwF$_{\mathcal{VC}}^{P \mathcal{H}_*^?}$.*

*Proof.* See theorem 4.8 in [4] for $\mathcal{B}$, proposition 2.3 in [4] and fact 3.1 in [6] for $\mathcal{N}$ and theorem 7.1 in [5] for $\mathcal{L}$. □

**Theorem 2.2.** (1). *$\mathcal{B}$ implies that the FICwF$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for the following groups.*

*(a). Finitely generated virtually polycyclic groups.*

*(b). Finitely generated virtually solvable subgroup of $GL_n(\mathbb{C})$.*

*In addition if we assume $\mathcal{L}$ then we can remove the condition 'finitely generated' from (a) and (b).*

(2). *$\mathcal{B}$ and $\mathcal{N}$ together imply that the FICwF$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for the following groups.*

*(a). Cocompact discrete subgroups of virtually connected Lie groups.*

*(b). Artin full braid groups.*

(3). *$\mathcal{B}$, $\mathcal{N}$ and $\mathcal{L}$ imply that the FICwF$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for the following groups.*

*(a). Fundamental groups of Seifert fibered and Haken 3-manifolds. And fundamental groups of arbitrary 3-manifolds if we assume the Geometrization conjecture.*

*(b). Weak strongly poly-surface groups (see Definition 4.1).*

*(c). Extensions of closed surface groups by surface groups.*

*(d). $\pi_1(M) \rtimes \mathbb{Z}$, where $M$ is a closed Seifert fibered space.*

The following Corollary follows using ($a$) of Lemma 3.4.

**Corollary 2.1.** *Let $G$ be a group containing a finite index subgroup $\Gamma$. Assume that $\Gamma$ is isomorphic to a group appearing in Theorem 2.2 together with the corresponding hypotheses. Then the FICwF$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for $G$.*

Theorem 2.1 and 2.2 together imply the following corollary.

**Corollary 2.2.** *The FICwF$_{\mathcal{VC}}^{P \mathcal{H}_*^?}$ is true for groups appearing in Theorem 2.2.*

Corollary 2.1 together with Proposition 1.1 and proposition 4.10 in [9] imply the following for the Isomorphism Conjecture in the algebraic $K$-theory case.

**Corollary 2.3.** *Let $G$ be a group which contains a finite index subgroup $\Gamma$ so that either $\Gamma$ is a group appearing in Theorem 2.2 or assume that the $FICwF_{\mathcal{VC}}^{P\mathcal{H}_*^?}$ is true for $\Gamma$. Then for $R = \mathbb{Z}$ the algebraic $K$-theory assembly map*

$$\mathcal{H}_n^{G \wr H}(E_{\mathcal{VC}}(G \wr H), \mathbf{K}_R) \to \mathcal{H}_n^{G \wr H}(pt, \mathbf{K}_R)$$

*is an isomorphism for $n \leq 1$, where $H$ is a finite group. Or equivalently the $ICwF_{\mathcal{VC}}^{K\mathcal{H}_*^?}$ is true for $G$ in dimension $\leq 1$ when $R = \mathbb{Z}$.*

*Here recall that the above map is induced by the map $E_{\mathcal{VC}}(G \wr H) \to pt$ and $\mathcal{H}_n^?(X, \mathbf{K}_R)$ is the standard notation for $_K\mathcal{H}_n^?(X)$.*

**Remark 2.1.** $\mathcal{L}$ is known for many cases, for example see theorem 7.1 in [5] for the $FIC_{\mathcal{VC}}^{K\mathcal{H}_*^?}$ and $FIC_{\mathcal{VC}}^{L\mathcal{H}_*^?}$ and for the $FIC_{\mathcal{VC}}^{KH\mathcal{H}_*^?}$ see theorem 11.4 in [1]. We recall now some results related to $\mathcal{N}$. In [2], $IC_{\mathcal{VC}}^{K\mathcal{H}_*^?}$ is proved for the fundamental groups of closed strictly negatively curved Riemannian manifolds. We have already mentioned in the abstract that the proof of the $FIC_{\mathcal{VC}}^{P\mathcal{H}_*^?}$ is known for all the groups appearing in Theorem 2.2 except for those in $3(c)$ and $3(d)$ (the precise references are given during the proof). In case $3(a)$ $FICwF_{\mathcal{VC}}^{P\mathcal{H}_*^?}$ is also known. The main ingredients behind the proof of Theorem 2.2 are Lemmas 3.3 to 3.6.

**Remark 2.2.** We should remark that the $FIC_{\mathcal{VC}}^{P\mathcal{H}_*^?}$ was proved for certain class of mapping class groups of surfaces of lower genus in [3]. Assuming $\mathcal{B}$, $\mathcal{N}$ and $\mathcal{L}$ and using the methods of this article the $FICwF_{\mathcal{VC}}^{\mathcal{H}_*^?}$ can also be deduced for these mapping class groups. We leave the details to the reader.

## 3. Some preliminary results

**Lemma 3.1.** *[algebraic lemma, [6]] Let $K$ be a finite index normal subgroup of a group $G$, then $G$ embeds in $K \wr (G/K)$.*

The following lemma is standard.

**Lemma 3.2.** *(a). Let $G$ be a finite index subgroup of a group $K$. Then there is a subgroup $G_1 < G$ which is normal and of finite index in $K$.*

*(b). In addition if $K$ is finitely presented then there is a subgroup $G_1 < G$ which is characteristic and of finite index in $K$.*

**Lemma 3.3.** $\mathcal{B}$ *implies that the $\mathcal{P}_{\mathcal{VC}}^{\mathcal{H}_*^?}$-property is satisfied.*

*Proof.* Note that for two virtually cyclic groups $C_1$ and $C_2$, $C_1 \times C_2$ contains a free abelian (of rank $\leq 2$) normal subgroup (say $A$) of finite index. Hence by Lemma 3.1 $C_1 \times C_2$ is a subgroup of $A \wr F$ for some finite group $F$. Therefore $\mathcal{B}$ and the hereditary property of the FICwF$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ completes the proof.                                                     $\square$

**Lemma 3.4.** *Assume that the $\mathcal{P}_{\mathcal{VC}}^{\mathcal{H}_*^?}$-property is satisfied.*

*(a). Let $G$ be a finite index subgroup of a group $K$. If the FICwF$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for $G$ then the FICwF$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is also true for $K$.*

*(b). Let $p : G \to Q$ be a surjective group homomorphism and assume that the FICwF$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for $Q$, for $\ker(p)$ and for $p^{-1}(C)$ for any infinite cyclic subgroup $C$ of $Q$. Then $G$ satisfies the FICwF$_{\mathcal{VC}}^{\mathcal{H}_*^?}$.*

*Proof.* Note that $(a)$ is same as $(2)$ of proposition 5.2 in [15] with the only difference that there we assumed $G$ is also normal in $K$. But this normality condition can be removed using $(a)$ of Lemma 3.2 and the hereditary property and then applying $(2)$ of proposition 5.2 in [15].

Also $(b)$ is easily deduced from $(a)$ and $(3)$ of proposition 5.2 in [15]. Since the hypothesis of $(3)$ of proposition 5.2 in [15] was that the FICwF$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for $Q$ and for $p^{-1}(C)$ for any virtually cyclic subgroup (including the trivial group) $C$ of $Q$. And by definition a virtually cyclic group is either finite or contains an infinite cyclic subgroup of finite index.                                                     $\square$

An immediate Corollary of $(a)$ is the following.

**Corollary 3.1.** *If the $\mathcal{P}_{\mathcal{VC}}^{\mathcal{H}_*^?}$-property is satisfied then the FICwF$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for all $H \in \mathcal{VC}$.*

**Lemma 3.5.** *Assume $\mathcal{N}$ and that the $\mathcal{P}_{\mathcal{VC}}^{\mathcal{H}_*^?}$-property is satisfied. Let $M$ be one of the following manifolds.*

*(a). A closed Haken 3-manifold such that there is a hyperbolic piece in the JSJT decomposition of the manifold.*

*(b). A compact irreducible 3-manifold with nonempty incompressible boundary and there is at least one torus boundary component.*

*(c). A compact surface.*

*Then the FICwF$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for $\pi_1(M)$.*

*Proof.* In $(a)$ by theorem 3.2 in [8] $M$ supports a nonpositively curved metric. The proof is now immediate.

For $(b)$ let $N$ be the double of $M$ along boundary components of genus $\geq 2$. Then we can again apply theorem 3.2 in [8] to see that the interior of $N$ supports a complete nonpositively curved metric so

that near the boundary the metric is a product of flat tori and $(0, \infty)$. Therefore the double of $N$ is a closed nonpositively curved manifold. Now we can apply the hereditary property to complete the proof.

The proof of $(c)$ also follows by taking the double and using the fact that a closed surface either has finite fundamental group or supports a nonpositively curved metric. $\square$

**Lemma 3.6.** *Assume $\mathcal{N}$, $\mathcal{L}$ and that the $\mathcal{P}_{\mathcal{VC}}^{\mathcal{H}_*^?}$-property is satisfied. Then the followings hold.*

*(a). The $FICwF_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for countable free groups.*

*(b). If the $FICwF_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for two countable groups $G_1$ and $G_2$ then the $FICwF_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for the free product $G_1 * G_2$.*

We recall here that the above lemma was proved in proposition 5.3 and lemma 6.3 in [15] under a different hypothesis.

*Proof.* For the proof of $(a)$ note that a finitely generated free group is isomorphic to the fundamental group of a compact surface and hence $(c)$ of Lemma 3.5 applies. And since an infinitely generated countable free group is a direct limit of finitely generated free subgroups we can apply $\mathcal{L}$ in addition to $(c)$ of Lemma 3.5 to complete the proof.

For the proof of $(b)$ consider the following short exact sequence.

$$1 \to K \to G_1 * G_2 \to G_1 \times G_2 \to 1,$$

where $K$ is the kernel of the homomorphism $p : G_1 * G_2 \to G_1 \times G_2$. By (1) of proposition 5.2 in [15] the $FICwF_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for the product $G_1 \times G_2$. Now by lemma 5.2 in [13] $p^{-1}(C)$ is a countable free group if $C$ is either the trivial group or an infinite cyclic subgroup of $G_1 \times G_2$. Therefore we can apply the previous assertion and $(b)$ of Lemma 3.4 to the above exact sequence to complete the proof. $\square$

## 4. Proof of Theorem 2.2

At first recall that by Lemma 3.3 $\mathcal{B}$ implies that the $\mathcal{P}_{\mathcal{VC}}^{\mathcal{H}_*^?}$-property is satisfied. Therefore we can use Lemma 3.4, Corollary 3.1, Lemma 3.5 and Lemma 3.6 in the proof of the Theorem.

*Proof of 1(a).* Let $\Gamma$ be a finitely generated virtually polycyclic group. That is, $\Gamma$ contains a (finitely generated) polycyclic subgroup of finite index. Also a polycyclic group contains a poly-$\mathbb{Z}$ subgroup of finite index (see 5.4.15 in [11]). Therefore by $(a)$ of Lemma 3.4 it is enough to prove the $FICwF_{\mathcal{VC}}^{\mathcal{H}_*^?}$ for poly-$\mathbb{Z}$ groups. Hence we assume that $\Gamma$ is a finitely generated poly-$\mathbb{Z}$ group. Now the proof is by induction on the virtual cohomological dimension $(vcd)$ of $\Gamma$.

We need the following lemma which is easily deduced from lemma 4.1 and lemma 4.4 in [4].

**Lemma 4.1.** *Let k=vcd($\Gamma$). Then there is a short exact sequence of groups.*

$$1 \to \Gamma' \to \Gamma \to A \to 1,$$

*where $\Gamma'$ and A satisfy the following properties.*
*(a). $\Gamma'$ is a poly-$\mathbb{Z}$ group with vcd($\Gamma'$)$\leq k-2$.*
*(b). There is a short exact sequence.*

$$1 \to A' \to A \to B \to 1,$$

*where $A'$ contains a finitely generated free abelian subgroup of finite index and B is either trivial or infinite cyclic or the infinite dihedral group $D_\infty$.*

Now assume that the FICwF$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for finitely generated poly-$\mathbb{Z}$ groups of virtual cohomological dimension $\leq k-1$. Then apply the hypothesis and (b) of Lemma 3.4 to the two exact sequences in Lemma 4.1. The details arguments are routine and we leave it to the reader. This completes the proof of 1(a).

*Proof of 1(b).* Let $\Gamma$ be a finitely generated virtually solvable subgroup of $GL_n(\mathbb{C})$. Then $\Gamma$ contains a finitely generated torsion free solvable subgroup of finite index. Hence by (a) of Lemma 3.4 we can assume that $\Gamma$ is torsion free. By a theorem of Mal'cev ( see 15.1.4 in [11]) there are subgroups $\Gamma_2 < \Gamma_1 < \Gamma$ with the following properties.

- $\Gamma_2$ and $\Gamma_1$ are normal in $\Gamma$.
- $\Gamma_2$ is nilpotent.
- $\Gamma_1/\Gamma_2$ is abelian.
- $\Gamma/\Gamma_1$ is finite.

Therefore we have the following exact sequences.

$$1 \to \Gamma_1/\Gamma_2 \to \Gamma/\Gamma_2 \to \Gamma/\Gamma_1 \to 1.$$
$$1 \to \Gamma_2 \to \Gamma \to \Gamma/\Gamma_2 \to 1.$$

Since $\Gamma$ is finitely generated and $\Gamma/\Gamma_1$ is finite, it follows that $\Gamma_1$ is also finitely generated. Hence $\Gamma_1/\Gamma_2$ is finitely generated abelian. Consequently $\Gamma/\Gamma_2$ contains a finitely generated free abelian subgroup of finite index. Since $\Gamma_2$ is torsion free nilpotent it is poly-$\mathbb{Z}$. Now using 1(a), $\mathcal{B}$ and (b) of Lemma 3.4 we complete the proof.

*Proof of 2(a).* We follow the structure of the proof of theorem 2.1 in [4]. So let $\Gamma$ be a discrete cocompact subgroup of a virtually connected Lie group $G$. First let us show that we can assume that the Lie group has no nontrivial compact connected normal subgroup. So let $C$ denotes

the maximal compact connected normal subgroup of $G$. Let $\Gamma' = q(\Gamma)$ and $F = \Gamma \cap C$ where $q : G \to G/C$ is the quotient map. Then $F$ is a finite normal subgroup of $\Gamma$ with quotient $\Gamma'$ which is a discrete cocompact subgroup of the virtually connected Lie group $G/C$. Note that $G/C$ contains no nontrivial compact connected normal subgroup. Applying $(b)$ of Lemma 3.4 to the map $q$ we see that the $\mathrm{FICwF}_{\mathcal{VC}}^{\mathcal{H}_*^?}$ for $\Gamma'$ implies the $\mathrm{FICwF}_{\mathcal{VC}}^{\mathcal{H}_*^?}$ for $\Gamma$. Since if a group contains a finite normal subgroup with infinite cyclic quotient then the group is virtually infinite cyclic.

Therefore from now on we assume that $G$ has no nontrivial compact connected normal subgroup. Next we deduce that we can also assume that the identity component $G_e$ of $G$ is a semisimple Lie group. By the Levi decomposition we have that $G$ contains a maximal closed connected normal solvable subgroup $R$ with quotient $S$ which is a virtually connected Lie group with the identity component $S_e$ semisimple. That is we have the following exact sequence.

$$1 \to R \to G \to S \to 1.$$

Let $\Gamma_R = \Gamma \cap R$ and $\Gamma_S = p(\Gamma)$, where $p$ denotes the homomorphism $G \to S$. Then we have the following (see 2.6 (a), (b) and (c) in [4]). A short exact sequence

$$1 \to \Gamma_R \to \Gamma \to \Gamma_S \to 1,$$

where $\Gamma_R$ is virtually poly-$\mathbb{Z}$ and discrete cocompact in $R$ and $\Gamma_S$ is discrete cocompact in $S$. Therefore using case $1(a)$ of the Theorem and by $(b)$ of Lemma 3.4 we see that it is enough to prove the $\mathrm{FICwF}_{\mathcal{VC}}^{\mathcal{H}_*^?}$ for $\Gamma_S$.

Henceforth we assume that $G$ has no nontrivial compact connected normal subgroup and $G_e$ is semisimple. Let $H = \{g \in G \mid gh = hg$ for all $h \in G_e\}$ and $q : G \to G/H$ be the quotient map. Then we have the following (see 2.5 (a), (b), (c) and (d) in [4]).

$$1 \to \Gamma_H = H \cap \Gamma \to \Gamma \to \Gamma' = q(\Gamma) \to 1,$$

where $\Gamma_H$ is a finite extension of a finitely generated abelian group, $\Gamma'$ is a cocompact discrete subgroup of $G/H$ and $G/H$ is a virtually connected linear Lie group such that the identity component of $G/H$ is semisimple. Now note that $\Gamma'$ acts cocompactly and properly discontinuously on a complete nonpositively curved Riemannian manifold (namely on $(G/H)/C$, where $C$ is maximal compact subgroup of $G/H$). Hence by $\mathcal{N}$ the $\mathrm{FICwF}_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for $\Gamma'$. Since $\Gamma_H$ contains a finitely generated free abelian subgroup of finite index, we can therefore apply

$1(a)$ of the Theorem and Lemma 3.4 to the above exact sequence to complete the proof of $2(a)$.

*Proof of 2(b).* Let
$$M_n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{C}^{n+1} \mid x_i \neq x_j, \text{ for } i \neq j\}.$$
Then the symmetric group $S_{n+1}$ acts on $M_n$ freely. By definition the Artin full braid group (denoted by $B_n$) on $n$ strings is the fundamental group of the manifold $M_n/S_{n+1}$ and the pure braid group is defined as the fundamental group of $M_n$ and is denoted by $PB_n$. Hence $PB_n$ is a normal subgroup of $B_n$ with quotient $S_{n+1}$. Using $(a)$ of Lemma 3.4 we see that it is enough to prove the FICwF$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ for $PB_n$. The proof is by induction on $n$. For $n = 1$, $PB_n \simeq \mathbb{Z}$ and hence by Corollary 3.1 we can start the induction. So assume the result for $PB_k$ for $k \leq n-1$. We will show that the FICwF$_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for $PB_n$. Consider the following projection map.
$$p : M_n \to M_{n-1}$$
$$(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n).$$
This map is a locally trivial fiber bundle projection with fiber diffeomorphic to $N = \mathbb{C} - \{n\text{-points}\}$. Therefore we have the following exact sequence.
$$1 \to \pi_1(N) \to PB_n \to PB_{n-1} \to 1.$$
The following is easy to show.

- For an infinite cyclic subgroup $C$ of $PB_{n-1}$, $p_*^{-1}(C)$ is isomorphic to $\pi_1(N_f)$ where $N_f$ denotes the mapping torus of the monodromy diffeomorphism $f : N \to N$ corresponding to a generator of $C$. It follows that $N_f$ is diffeomorphic to the interior of a compact Haken 3-manifold with incompressible boundary components.

Note that all the boundary components of $N_f$ could be Klein bottles. In that case we can take a finite sheeted cover of $N_f$ which is a Haken 3-manifold with incompressible tori boundary components.

Now we can apply $(a)$ and $(b)$ of Lemma 3.4, $(b)$ of Lemma 3.5 and $(a)$ of Lemma 3.6 to complete the proof of $2(b)$.

*Proof of 3(a).* The proof goes in the same line as the proof of the FICwF$_{\mathcal{VC}}^{P\mathcal{H}_*^?}$ for 3-manifold groups (see [13] and [14]).

First of all using $\mathcal{L}$ we can assume that the 3-manifold is compact. Also applying $(a)$ of Lemma 3.4 we can assume that the 3-manifold is orientable and that all boundary components are orientable. So let $M$ be a compact orientable 3-manifold with orientable boundary. If there is any sphere boundary then we can cap these boundaries

by 3-discs. This does not change the fundamental group. Therefore we can assume that either $M$ is closed or all boundary components are orientable surfaces of genus $\geq 1$. By lemma 3.1 in [13] $\pi_1(M) \simeq \pi_1(M_1) * \cdots * \pi_1(M_k) * F^r$, where $M_i$ is a compact orientable irreducible 3-manifold for each $i = 1, 2, \ldots, k$ and $F^r$ is a free group of rank $r$. Hence applying Lemma 3.6 we are reduced to the situation of compact orientable irreducible 3-manifold $M$. Also we can assume $\pi_1(M)$ is infinite as the $\text{FICwF}_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for finite groups.

Now by Thurston's Geometrization conjecture $M$ has the following possibilities.

- Closed Seifert fibered space.
- Haken manifold.
- Closed hyperbolic manifold.

Using $\mathcal{N}$ we only have to consider the first two cases. In the first case there is an exact sequence.

$$1 \to \mathbb{Z} \to \pi_1(M) \to \pi_1^{orb}(S) \to 1,$$

where $S$ is the base orbifold of the Seifert fibered space $M$. It is well known that $\pi_1^{orb}(S)$ is either finite or contains a closed surface subgroup of finite index. Now we can apply $(c)$ of Lemma 3.5 and $(a)$ and $(b)$ of Lemma 3.4 to complete the proof in this case.

For the Haken manifold situation let us first consider the nonempty boundary case. By lemma 6.4 in [13] we have that $\pi_1(M) \simeq \pi_1(M_1) * \cdots * \pi_1(M_l) * F^s$, where each $M_i$ is compact orientable irreducible and has incompressible boundary and $F^s$ is a free group of rank $s$. We apply $(a)$ and $(b)$ of Lemma 3.6, $(b)$ of Lemma 3.5, theorem 1.1.1 in [14], $\mathcal{N}$ and the hereditary property of the $\text{FICwF}_{\mathcal{VC}}^{\mathcal{H}_*^?}$ to complete the proof in this case. (Recall here that the theorem 1.1.1 and the remark 1.1.1 in [14] say that the fundamental group of each $M_i$ is a subgroup of the fundamental group of a closed 3-manifold $P$ which is either Seifert fibered or supports a nonpositively curved metric.)

Finally we assume that $M$ is a closed Haken manifold. If $M$ contains no incompressible torus then by Thurston's hyperbolization theorem $M$ is hyperbolic. Hence using $\mathcal{N}$ and $(a)$ of Lemma 3.5 we can assume that $M$ contains an incompressible torus and has no hyperbolic piece in the JSJT decomposition. Therefore $M$ is a graph manifold. By theorem 1.1 in [10] either $M$ has a finite sheeted cover which is a torus bundle over the circle or for any positive integer $k$ there is a finite sheeted cover $M_k$ of $M$ so that rank of $H_1(M_k, \mathbb{Z})$ is $\geq k$. In the first possibility we can apply $1(a)$ of the Theorem to complete the proof. For the second

case let $k \geq 2$ and consider the following exact sequence.

$$1 \rightarrow [\pi_1(M_k), \pi_1(M_k)] \rightarrow \pi_1(M_k) \rightarrow H_1(M_k, \mathbb{Z}) \rightarrow 1.$$

By $(a)$ of Lemma 3.4 it is enough to prove the $\mathrm{FICwF}_{\mathcal{VC}}^{\mathcal{H}_*^?}$ for $\pi_1(M_k)$. We will apply $(b)$ of Lemma 3.4 to the above exact sequence. So Let $C$ be either the trivial group or an infinite cyclic subgroup of $H_1(M_k, \mathbb{Z})$. Then $A^{-1}(C) \simeq \pi_1(K)$ where $K$ is a noncompact irreducible (being an infinite sheeted covering of an irreducible 3-manifold) 3-manifold. Here $A$ denotes the homomorphism $\pi_1(M_k) \rightarrow H_1(M_k, \mathbb{Z})$. Let $K = \cup_i K_i$ where $K_i$'s are increasing union of non-simply connected compact submanifold of $K$. By lemma 6.3 and lemma 6.4 in [13] we have that $\pi_1(K_i) \simeq \pi_1(M_1) * \cdots * \pi_1(M_l) * F^s$, where each $M_i$ is compact orientable irreducible and has incompressible boundary and $F^s$ is a free group of rank $s$. Therefore we can use $\mathcal{L}$, Lemma 3.6 and Haken manifold with nonempty boundary case as in the previous paragraph to complete the proof of $2(b)$.

*Proof of 3(b).* Let us recall the definition of weak strongly poly-surface group.

**Definition 4.1.** *[definition 1.2.1 in [14]]* A discrete group $\Gamma$ is called *weak strongly poly-surface* if there exists a finite filtration of $\Gamma$ by subgroups: $1 = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n = \Gamma$ such that the following conditions are satisfied:

- $\Gamma_i$ is normal in $\Gamma$ for each $i$.
- $\Gamma_{i+1}/\Gamma_i$ is isomorphic to the fundamental group of a surface $F_i$ (say).
- for each $\gamma \in \Gamma$ and $i$ there is a diffeomorphism $f : F_i \rightarrow F_i$ such that the induced automorphism $f_\#$ of $\pi_1(F_i)$ is equal to $c_\gamma$ up to inner automorphism, where $c_\gamma$ is the automorphism of $\Gamma_{i+1}/\Gamma_i \simeq \pi_1(F_i)$ induced by the conjugation action on $\Gamma$ by $\gamma$.

In such a situation we say that the group $\Gamma$ has *rank* $\leq n$.

The proof is by induction on the rank $n$ of the weak strongly poly-surface group $\Gamma$. Therefore assume that the $\mathrm{FICwF}_{\mathcal{VC}}^{\mathcal{H}_*^?}$ is true for all weak strongly poly-surface group of rank $\leq n-1$ and let $\Gamma$ has rank $n$. Consider the following exact sequence.

$$1 \rightarrow \Gamma_1 \rightarrow \Gamma \rightarrow \Gamma/\Gamma_1 \rightarrow 1.$$

The followings are easy to verify.

- $\Gamma/\Gamma_1$ is weak strongly poly-surface and has rank $\leq n-1$.

- $q^{-1}(C)$ is either a surface group or isomorphic to the fundamental group of a 3 manifold, where $C$ is either the trivial group or an infinite cyclic subgroup of $\Gamma/\Gamma_1$ respectively.

Now we can apply the induction hypothesis, $(b)$ of Lemma 3.4, $(c)$ of Lemma 3.5 and $3(a)$ of the Theorem to show that the FICwF$_{\mathcal{VC}}^{\mathcal{H}^?_*}$ is true for $\Gamma$.

*Proof of 3(c).* Let $\Gamma$ be an extension of a closed surface group by a surface group. Hence we have the following.

$$1 \to \pi_1(S) \to \Gamma \to \pi_1(S') \to 1$$

where $S$ is a closed surface and $S'$ is a surface.

Using $(a)$ of Lemma 3.4 and $(c)$ of Lemma 3.5 we can assume that $\pi_1(S')$ is not finite (and hence torsion free). Also since extension of finite group by infinite cyclic group is virtually cyclic we can also assume that $\pi_1(S)$ is infinite (and hence again torsion free). Recall that an automorphism of a closed surface group is induced by a diffeomorphism of the surface. Hence for any infinite cyclic subgroup $C$ of $\pi_1(S')$, $p^{-1}(C)$ is isomorphic to the fundamental group of a 3-manifold. Therefore we can apply $(b)$ of Lemma 3.4 and $3(a)$ of the Theorem to complete the proof of $3(c)$.

*Proof of 3(d).* Let $\Gamma = \pi_1(M) \rtimes \mathbb{Z}$, where $M$ is a closed Seifert fibered space. Using $(b)$ of Lemma 3.2 and $(a)$ of Lemma 3.4 we can assume that $M$ and its base surface $S$ are both orientable. This implies that the cyclic subgroup (say $C$) of $\pi_1(M)$ generated by a regular fiber is central (see [7]). Now if $\pi_1^{orb}(S)$ is finite then $\Gamma$ is virtually poly-$\mathbb{Z}$. Therefore using case $1(a)$ of the Theorem we can assume that $\pi_1^{orb}(S)$ is infinite and hence $C$ is also infinite cyclic. Since $C$ is central we have the following exact sequence.

$$1 \to \mathbb{Z} \to \pi_1(M) \rtimes \mathbb{Z} \to \pi_1^{orb}(S) \rtimes \mathbb{Z} \to 1.$$

As $\pi_1^{orb}(S)$ is infinite, it contains a closed surface subgroup of finite index. Now using $(b)$ of Lemma 3.2 it is easy to find a characteristic closed surface subgroup of $\pi_1^{orb}(S)$ of finite index. Therefore $\pi_1^{orb}(S) \rtimes \mathbb{Z}$ contains a 3-manifold group of finite index. Now we can apply $(a)$ and $(b)$ of Lemma 3.4 and $1(a)$ and $3(a)$ of the Theorem to complete the proof of $3(d)$.

This completes the proof of Theorem 2.2.

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